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Exact densities of states of fixed trace ensembles of random matrices

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Abstract. The densities of states and the associated characteristic functions of fixed-trace ensembles (FTEs) of $N \times N$ random matrices M_N, ($\mathrm{tr}(M_N^+ M_N) = \text{constant}$), are calculated exactly at finite N in the case of real-symmetric matrices and of Hermitian matrices. The exact radial density is calculated at finite N for fixed-trace ensembles of complex matrices with no further restrictions on the entries. The density calculated in the Hermitian case coincides with very recent literature results. The exact finite-N density of states of any ensemble of $N \times N$ random matrices of a given symmetry, whose probability density depends only on $\mathrm{tr}(S_N^+ S_N)$, is simply obtained from the density of the FTE of the same symmetry by a one-dimensional integral.

1. Introduction

Random matrix theory (RMT), which has been for decades, and is yet, of primary importance in multivariate statistical analysis [1], has also found significant uses in various other fields [2, 3]. It finds, for instance, considerable use in various branches of physics, notably in nuclear physics, quantum chaology and for investigating Hamiltonians of disordered and strongly interacting quantum systems [2, 3] and references therein.

Most applications of RMT in physics deal with large-N matrices. Among the ensembles of $N \times N$ random matrices S_N, three Gaussian ensembles have been studied extensively [2, 3] and are still investigated [4]. Their probability densities are proportional to $\exp(-\,\mathrm{tr}(S_N^2))$ where tr means trace. Matrices are real symmetric for the Gaussian orthogonal ensemble (GOE), Hermitian for the Gaussian unitary ensemble (GUE) and quaternion self-dual for the Gaussian symplectic ensemble (GSE) [3].

Besides applications of large-N matrices, there are physical investigations, notably of disordered solids, which benefit or might benefit from a knowledge of exact properties of ensembles of random matrices at finite N, most often $N = 2, 3$. Many properties of interest in concentrated crystalline alloys, glasses, nanostructured materials, quasicrystals are, for instance, either represented by second-rank tensors in 3D or involve them. A non-exhaustive list includes electric field gradients at nuclei of many isotopes that are investigated by NMR, NQR, PAC and Mössbauer spectroscopy or are modelled ([5–10] and references therein), spin Hamiltonian parameters that are investigated by EPR [11], demagnetizing tensor [12], magnetic dipolar tensors and fields [13, 14], atomic level stresses in models of disordered solids [15, 16] and dispersion tensor in homogeneous porous media [17, 18]. Many papers among those quoted above use, explicitly or implicitly, characteristics established for random matrix ensembles (RMEs) while other published results might as well be derived from them (see [10] and references therein for the case of electric field gradients). The zero probability of finding an electric field gradient with an asymmetry parameter equal to zero in many disordered structures originates from correlations due to the Jacobian [7, 10] as does the level-repulsion found in RMT [2,3]. The distributions of local atomic stresses in models of amorphous solids as given in appendix B of [15] are, for instance, directly obtained from the GOE with $N = 3$ [16]. Moreover, new investigations might profit from a comparison with general results established for finite-N RMEs. For instance, distributions of characteristics of the dispersion tensor and not only average properties might be investigated in the case of convection–diffusion phenomena. A final illustrative example is that of electric field gradients produced by a random distribution of unscreened point defects in cubic solids [8]. The distribution of the electric field gradient tensor, represented here by a 3×3 matrix V with $\mathrm{tr}\, V = 0$, is $p(V) \propto \exp(-c^2\, \mathrm{tr}(V^2))$ for large defect concentrations while it is $p(V) \propto (a^2+b^2\, \mathrm{tr}(V^2))^{-3}$ for small defect concentrations [8]. A calculation of characteristics of RMEs, named spherical ensembles in [19], whose probability densities, $g(\mathrm{tr}(S_N^+ S_N))$, depend only on $\mathrm{tr}(S_N^+ S_N)$ where S_N^+ is the Hermitian conjugate of S_N is thus of interest also for finite values of N.

Gaussian ensembles are notable members of the family of spherical ensembles [19]. They are indeed the only spherical ensembles which have independent matrix entries as deduced from the Porter–Rosenzweig theorem [3]. The latter theorem states indeed that ensembles of random matrices which are invariant under a change of basis and whose entries are statistically independent are Gaussian. Characteristics of spherical ensembles can be derived from those of fixed-trace ensembles, called hereafter FTEs, that were first defined by Rosenzweig and Bronk ([20, 21] as quoted in [3, ch 19]), by $\mathrm{tr}(S_N^+ S_N) =$ constant with no other constraint. The link between spherical ensembles and FTEs stems from the fact that FTEs are associated [19] with the uniform distribution of a random vector on an N_p-dimensional unit sphere surface whose properties are described in [22]. We define N_p as being the number of distinct real random variables which are necessary to construct a $N \times N$ matrix S_N of a given symmetry (real-symmetric, Hermitian, antisymmetric Hermitian, complex, etc). The stochastic representation (section 2) which links $N \times N$ random matrices G_N from Gaussian ensembles to $N \times N$ random matrices M_N of the FTE of the same symmetry is $G_N \stackrel{\mathrm{d}}{=} RM_N$, where $A \stackrel{\mathrm{d}}{=} B$ means that the random matrices A and B are identically distributed. The matrix M_N is associated with a vector $U^{(N_p)}$ uniformly distributed over the surface of the unit sphere in $I\mathbb{R}^{N_p}$ in the way described in section 2 while R^2 is χ^2 distributed with N_p degrees of freedom and is independent of $U^{(N_p)}$. Fixed-trace ensembles bear the same relationship to Gaussian ensembles that the microcanonical ensemble to the canonical ensembles in statistical physics [23]. Akemann *et al* [24] describe further interesting physical features of FTEs due to the interaction among eigenvalues introduced through a constraint. The previous arguments explain the interest of deriving exact characteristics of FTEs at finite N. The method we have used to derive them is based on the stochastic representation mentioned above which leads to integral relations between the characteristics of the Gaussian ensembles and of FTEs. These relations, expressed as Laplace transforms, give by inversion the expected densities of states. The present paper focuses on densities of states while other characteristics, for instance determinant distributions [25], have been or might be derived from similar methods.

Our purpose is thus to derive exact densities of states $\varrho_M(\lambda)$ for real-symmetric and Hermitian FTEs and the exact radial density $\varrho_M(r)$ for the complex FTE. Little before the very end of the present investigation which is a sequel of the previously quoted work [19], a paper of Akemann *et al* [24] was published on (generalized) restricted trace ensembles of Hermitian matrices which also include fixed-trace ensembles [3]. The exact density of states which we obtained independently for the unitary ensemble called here FTE(2) is identical with the one given by the latter authors who further derive the two-eigenvalue correlators [3, 26]. As our aims and the method used differ from those of [24], we still describe briefly the results for the unitary ensemble as it further exemplifies the type of calculations needed for the orthogonal ensemble. Theoretical densities of states of FTEs of real-symmetric matrices and of Hermitian matrices and radial densities of complex random matrices are compared with results of Monte Carlo simulations.

2. The ingredients of the method

For real-symmetric matrices and for Hermitian matrices, the number of distinct real random variables N_p is given by

$$N_m = N(N-1)/2 \qquad N_p = N + \beta N_m \tag{2.1}$$

with the parameter $\beta = 1, 2$ respectively [2, 3]. The distinct elements needed to construct the considered $N \times N$ matrices H_N are indeed $(i, j = 1, \ldots, N)\{H_{ij}, (j \geqslant i)\}$ for $\beta = 1$ and $\{H_{ii}, \mathrm{Re}\,(H_{ij}), \mathrm{Im}\,(H_{ij}), (j > i)\}$ for $\beta = 2$ while they are $\{\mathrm{Re}\,(H_{ij}), \mathrm{Im}\,(H_{ij})\}$ for complex matrices ($N_p = 2N^2$). After having presented some classical results on Gaussian ensembles [3] from which we will derive the FTEs densities of states in the next section, we establish general relations between the densities of states of spherical RMEs of a given symmetry and that of the FTE of the same symmetry. To simplify the notations in the following, $X(\beta)$ or X_β means that symbol X is associated with ensembles of real-symmetric ($\beta = 1$) matrices or of Hermitian matrices ($\beta = 2$) while $X(c)$ or X_C is used for ensembles of complex $N \times N$ matrices with no further restrictions on the entries. Moreover, $I_{[a,b]}(x)$ is an indicator function whose value is 1 if $a \leqslant x \leqslant b$ and 0 otherwise.

2.1. Gaussian ensembles

A random variable whose distribution is Gaussian with a zero mean and a variance σ^2 is hereafter denoted as $N(0, \sigma^2)$. For the GOE and the GUE, the N_p variables which constitute the matrix elements S_{ij} are independent $N(0, \sigma^2(1 + \delta_{ij})/2)$ variables ($i = 1, \ldots, N$, $j = i, \ldots, N$). The GO(U)E probability density function is then

$$g_\beta(S_N) = K_N(\beta) \exp(-\,\mathrm{tr}(S_N^2)/(2\sigma^2)) \tag{2.2}$$

where $K_N(\beta)$ is a normalization constant. Their average densities of states $\varrho_{G\beta}(\lambda)$ are given by Mehta [3] for variances $\sigma^2 = 1/\beta$.

(1) For the GUE ($\beta = 2, \sigma = 1/\sqrt{2}$):

$$\varrho_{G2}(\lambda) = \frac{1}{N} \sum_{j=0}^{N-1} \varphi_j^2(\lambda) \tag{2.3}$$

with

$$\varphi_j(\lambda) = (2^j j!\sqrt{\pi})^{-1/2} \exp(-\lambda^2/2) H_j(\lambda) \tag{2.4}$$

where $H_j(\lambda)$ is the Hermite polynomial of order j [3]. As shown by Ullah [27], the characteristic function $\phi_{G2}(t)$ of $\varrho_{G2}(\lambda)$ is simply

$$\phi_{G2}(t) = \langle \mathrm{e}^{\mathrm{i}\lambda t} \rangle_{G2} = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\lambda t} \varrho_{G2}(\lambda)\, \mathrm{d}\lambda = \frac{1}{N} \mathrm{e}^{-t^2/4} L_{N-1}^1\left(\frac{t^2}{2}\right) \tag{2.5}$$

where $L_{N-1}^1(x)$ is a Laguerre polynomial [28].

(2) For the GOE ($\beta = 1, \sigma = 1$), the density of states [3, 29] can be written as

$$\varrho_{G1}(\lambda) = \frac{1}{N}\left\{\sum_{j=0}^{N-1}\varphi_j^2(\lambda)\right\} + \frac{1}{2^{N+1}\Gamma(\frac{N}{2}+1)}\mathrm{e}^{-\lambda^2}H_{N-1}(\lambda)R_N(\lambda) \tag{2.6}$$

with

$$R_N(\lambda) = \sqrt{2}\exp\left(\frac{\lambda^2}{2}\right) - \sum_{m=0}^{[\frac{N}{2}]}\frac{H_{2m}(\lambda)}{2^{2m-1}m!} \tag{2.7}$$

if N is odd ($[x]$ stands for the largest integer less than or equal to x). Using equation (16) of [29], $R_N(\lambda)$ is similarly calculated when N is even:

$$R_N(\lambda) = \sqrt{2}\exp\left(\frac{\lambda^2}{2}\right)\mathrm{erf}\left(\frac{\lambda}{\sqrt{2}}\right) - \sum_{m=0}^{\frac{N}{2}-1}\frac{H_{2m+1}(\lambda)}{2^{2m}\Gamma(m+\frac{3}{2})}. \tag{2.8}$$

The characteristic function $\phi_{G_1}(t)$ [29], which involves infinite sums, is given by (2.9) after correcting some errors in equation (23) of [29]. Equation (2.9) is obtained from a Fourier transform of the GOE density of states (equation (11) of [29]) using integral 7.388.7 of [28]:

$$\phi_{G1}(t) = \frac{\mathrm{e}^{-t^2/4}}{N}L_{N-1}^1(t^2/2) - \frac{\mathrm{e}^{-t^2/4}}{2N}\left(\frac{\Gamma(\frac{N+1}{2})(N-1)!}{\Gamma(\frac{N}{2})}\right)^{\frac{1}{2}}$$
$$\times\sum_{k=0}^{\infty}\left(\frac{\Gamma(k+1+\frac{N}{2})}{\Gamma(k+\frac{N+3}{2})(2k+N+1)!}\right)^{\frac{1}{2}}\frac{(-1)^k}{2^k}t^{2k+2}L_{N-1}^{2k+2}\left(\frac{t^2}{2}\right). \tag{2.9}$$

It is possible to write (2.9) in a simpler way for $N = 2p+1$. From integrals 7.388.4 and 7.388.7 of [28], the Fourier transform of the density (equations (2.6) and (2.7)) is indeed

$$\phi_{G1}(t) = \mathrm{e}^{-t^2/4}\Bigg(\frac{L_{2p}^1(t^2/2)}{(2p+1)} - \frac{1}{2\Gamma(p+\frac{3}{2})}\sum_{k=0}^{p}\Gamma\left(p-k+\frac{1}{2}\right)\left(-\frac{t^2}{4}\right)^k$$
$$\times L_{2m-2k}^{2k}\left(\frac{t^2}{2}\right)\Bigg) + \mathrm{e}^{-t^2/2}\frac{\sqrt{\pi}(-1)^pH_{2p}(t)}{2^{2p+1}\Gamma(p+\frac{3}{2})}. \tag{2.10}$$

The empirical GO(U)E eigenvalue distribution function $F_N(\lambda) =$ (number of eigenvalues $\leqslant\lambda)/N$ tends asymptotically to a distribution whose probability density is $\varrho_W(\lambda)$ [29], the Wigner semicircle, whose scale parameter is its 'radius' a [3]:

$$\varrho_W(\lambda) = (2/\pi a^2)(a^2-\lambda^2)^{1/2}I_{[0,a^2]}(\lambda^2) \tag{2.11}$$

where a^2 is related to σ^2 (equation (2.2)) by $a^2 = 2\beta N\sigma^2$ (end of appendix A). Its characteristic function is

$$\phi_W(t) = \langle\mathrm{e}^{\mathrm{i}\lambda t}\rangle_W = \int_{-a}^{+a}\mathrm{e}^{\mathrm{i}\lambda t}\varrho_W(\lambda)\,\mathrm{d}\lambda = \frac{2}{at}J_1(at) \tag{2.12}$$

where $J_1(x)$ is a Bessel function of the first kind.

(3) The eigenvalues are complex for Gaussian complex $N\times N$ matrices with no further restrictions on the entries, that is with $N_p = 2N^2$. When the real and imaginary parts of all matrix elements are independent $N(0, \sigma^2/2)$ random variables, $g_c(S_N) \propto$

$\exp(-\operatorname{tr}(S_N S_N^+)/\sigma^2)$, the ensemble averaged fraction of eigenvalues located at a distance $r \leqslant R \leqslant r + \mathrm{d}r$ from the origin in the complex plane is $p_{GC}(r)\,\mathrm{d}r = 2\pi r \varrho_{GC}(r)\,\mathrm{d}r$ with

$$\varrho_{GC}(r) = \frac{1}{N\pi\sigma^2} \exp\left(-\frac{r^2}{\sigma^2}\right) \left\{ \sum_{k=0}^{N-1} \frac{(r/\sigma)^{2k}}{k!} \right\} \tag{2.13}$$

as calculated by Ginibre (see [3]). For $N \to \infty$ and $\sigma^2 = \sigma_G^2/N$, the density $\varrho_{GC}(r)$ is constant and equal to $1/(\pi\sigma_G^2)$ on a disc of radius σ_G.

2.2. *Spherical distributions of random vectors [22, 30–32]*

The distribution of a random N_p-dimensional vector $\boldsymbol{X}^{(N_p)}$ is called spherically symmetric [22, 31] or in short *spherical* [32] if its characteristic function is

$$\phi(\boldsymbol{t}) = \langle \exp(\mathrm{i}\boldsymbol{t} \cdot \boldsymbol{X}^{(Np)}) \rangle = \phi(t) \tag{2.14}$$

for all real N_p-dimensional vectors $\boldsymbol{t}$, where t is the modulus of $\boldsymbol{t}$, $t = [\sum_{k=1}^{N_p} t_k^2]^{1/2}$, or equivalently if it is invariant by any orthogonal transformation of $\mathrm{O}(N_p)$. Any spherical distribution of a random vector is in particular invariant by every permutation of its components. The density of $\boldsymbol{X}^{(N_p)}$ depends only on $\|\boldsymbol{X}^{(N_p)}\|$ whatever N_p, as its characteristic function depends only on the modulus t of $\boldsymbol{t}$ and conversely [22, 30–32]. For the spherical case, the Schoenberg theorem states that $\phi(t)$ is given by (2.14) if and only if it is represented by [22,30]

$$\phi(t) = \int_0^\infty \Omega_{N_p}(rt)\,\mathrm{d}P_{N_p}(r) \tag{2.15}$$

for some distribution $P_{N_p}(r)$, where $\Omega_{N_p}(t)$ is the characteristic function of a vector $\boldsymbol{U}^{(N_p)}$ which is uniformly distributed on the surface of the unit sphere in $I\!\mathbb{R}^{N_p}$ [22, 30–32]. An equivalent characterization of spherical random vectors is given by their stochastic representation [1, 22, 31, 32]:

$$\boldsymbol{X}^{(N_p)} \stackrel{\mathrm{d}}{=} R\boldsymbol{U}^{(N_p)} \tag{2.16}$$

where $\boldsymbol{A} \stackrel{\mathrm{d}}{=} \boldsymbol{B}$ means that the random vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are identically distributed and R is some non-negative random variable independent of $\boldsymbol{U}^{(N_p)}$. Some characteristics of the uniform distribution of $\boldsymbol{U}^{(N_p)}$ on the unit sphere surface are given in appendix A.

2.3. *Application to 'spherical' ensembles of random matrices*

To apply the results of the previous section to random matrices, it suffices to use the isomorphism between the space of $N \times N$ matrices and the Euclidean space $I\!\mathbb{R}^{N_p}$ as described in detail in [19] and as exemplified by (2.18) below.

Each matrix H_N is put in a one-to-one correspondence with a vector $\boldsymbol{X}^{(N_p)}$ of $I\!\mathbb{R}^{N_p}$, denoted $\boldsymbol{X}^{(N_p)} = \mathrm{Vect}(H_N)$, in such a way that the usual Euclidean norm of $\boldsymbol{X}^{(N_p)}$ is equal to the norm of H_N, $\|H_N\| = (\operatorname{tr}(H_N^+ H_N))^{1/2}$. The components of $\boldsymbol{X}^{(N_p)}$ are constructed from the distinct elements of H_N. A way, among many, of establishing a one-to-one correspondence between the distinct matrix elements of H_N and the components of $\boldsymbol{X}^{(N_p)}$ is first to form a matrix V_N whose diagonal elements are identical with those of H_N and whose off-diagonal elements are $\sqrt{2}$ times the off-diagonal elements of H_N for $\beta = 1, 2$ while $V_N = H_N$ in the complex case. The distinct elements of the columns of V_N are then stacked one under the other to form a single column. The components of $\boldsymbol{X}^{(N_p)}$ are thus $(V_{11}, V_{12}, V_{22}, V_{13}, V_{23}, V_{33}, \ldots)$ in the real symmetric case, $(V_{11}, \mathrm{Re}\,(V_{12}), \mathrm{Im}\,(V_{12}), V_{22}, \mathrm{Re}\,(V_{13}), \mathrm{Im}\,(V_{13}), \mathrm{Re}\,(V_{23}),$ $\mathrm{Im}\,(V_{23}), V_{33}, \ldots)$ in the Hermitian case and $(\mathrm{Re}\,(V_{11}), \mathrm{Im}\,(V_{11}), \mathrm{Re}\,(V_{21}), \mathrm{Im}\,(V_{21}), \ldots,$

$\mathrm{Re}\,(V_{N1})$, $\mathrm{Im}\,(V_{N1})$, $\mathrm{Re}\,(V_{12})$, $\mathrm{Im}\,(V_{12})$, $\mathrm{Re}\,(V_{22})$, $\mathrm{Im}\,(V_{22}), \ldots)$ in the complex case. The inverse transformation from $\boldsymbol{X}^{(N_p)}$ to H_N is denoted as $H_N = \mathrm{Mat}(\boldsymbol{X}^{(N_p)})$. The probability density of $S_N = \mathrm{Mat}(\boldsymbol{X}^{(N_p)})$, if it exists, is consequently $g(\mathrm{tr}(S_N^+ S_N))$ if $\boldsymbol{X}^{(N_p)}$ is a spherical N_p-dimensional vector as the Jacobian of the transformation from the probability distribution of the components of $\boldsymbol{X}^{(N_p)}$, which depends only on $\|\boldsymbol{X}^{(N_p)}\|$, to the probability distribution of the distinct matrix elements of H_N is a constant. By extension the latter RMEs have been named 'spherical' [19]. When the Mat transformation is applied to both sides of (2.16), it becomes

$$S_N \stackrel{\mathrm{d}}{=} R M_N. \tag{2.17}$$

The probability densities of the elements of S_N of a given symmetry are thus deduced to be mixtures of the probability density of the matrix elements of M_N, $M_N = \mathrm{Mat}(\boldsymbol{U}^{(N_p)})$, which has the same symmetry as S_N.

From vectors $\boldsymbol{U}^{(N_p)}$ uniformly distributed on the surface of the unit N_p-dimensional sphere with $N_p = 3, 4, 8$, we construct, for instance, the following FTEs $M_2(\beta)$ and $M_2(c)$ respectively:

$$M_2(1) = \begin{pmatrix} U(1) & U(2)/\sqrt{2} \\ U(2)/\sqrt{2} & U(3) \end{pmatrix},$$
$$M_2(2) = \begin{pmatrix} U(1) & (U(2) + \mathrm{i}U(3))/\sqrt{2} \\ (U(2) - \mathrm{i}U(3))/\sqrt{2} & U(4) \end{pmatrix}$$

and

$$M_2(c) = \begin{pmatrix} U(1) + \mathrm{i}U(2) & U(5) + \mathrm{i}U(6) \\ U(3) + \mathrm{i}U(4) & U(7) + \mathrm{i}U(8) \end{pmatrix} \tag{2.18}$$

where $U(k)$ is the kth component of the unit vector $\boldsymbol{U}^{(N_p)}(\sum_{k=1}^{N_p} U(k)^2 = 1 = \mathrm{tr}(M_N M_N^+))$. The distribution of $t = \mathrm{tr}(M_N(\beta))/\sqrt{N}(-1 \leqslant t \leqslant 1, \beta = 1, 2)$ is obtained in appendix A as $\frac{\Gamma(N_p/2)}{\sqrt{\pi}\Gamma((N_p-1)/2)}(1-t^2)^{\frac{N_p-3}{2}}$. We note the densities of states $\varrho_{M\beta}(\lambda)$ in the case of real eigenvalues (FTE(β)) while the radial density is named $\varrho_{MC}(r)$ in the case of the complex RME studied here (FTE(c)). The distinct elements of the matrices of the FTEs are not independent as they are related through $\mathrm{tr}(M_N M_N^+) = 1$ but their pair correlations are zero. The mixed moments $\langle\prod_{k=1}^{N_p} U(k)^{m_k}\rangle(m_k \geqslant 0)$ are indeed equal to zero if at least one of the m_k is odd as shown in [22, p 72]. The average of every FTEs matrix element and the pair correlation between two distinct matrix elements are thus zero as $\langle U(j)\rangle$ and pair correlations of the form $\langle U(j)U(k)\rangle$are zero whatever j, k with $j \neq k$.

2.4. The exact density of states of spherical ensembles

A transformation of both sides of (2.17), as done usually for Gaussian ensembles [3], from a probability distribution of matrix elements to a joint distribution of eigenvalues and the removal of identical factors in both members yields after integration over $N - k$ eigenvalues

$$\begin{aligned} \varrho_{S\beta}(\boldsymbol{v}_k) &= \int_L^\infty \varrho_{M\beta}(\boldsymbol{v}_k/r)\frac{\mathrm{d}F(r)}{r^k} \\ \boldsymbol{v}_k &= (\lambda_1, \lambda_2, \ldots, \lambda_k) \end{aligned} \tag{2.19}$$

whatever $1 \leqslant k \leqslant N$. In (2.19) $\mathrm{d}F(r)$ is the probability of finding a N_p-dimensional sphere of radius $r \leqslant R \leqslant r + \mathrm{d}r$. If we assume that there is a density $g(\mathrm{tr}(S_N^+ S_N))$ for the distribution of the considered spherical RME, then $\mathrm{d}F(r) = f(r)\,\mathrm{d}r$ where $f(r)$ is calculated to be

$$f(r) \propto r^{(N_p-1)} g(r^2) \tag{2.20}$$

by expressing distinct matrix elements in terms of spherical coordinates with $r^2 = \mathrm{tr}(S_N^+ S_N)$ and angles ($0 \leqslant \varphi_j \leqslant \pi$, $j = 1, \ldots, N_p - 2$, $0 \leqslant \varphi_{N_p-1} \leqslant 2\pi$).

The lower integration limit depends both on $F(r)$ and on $\boldsymbol{v}_k$. If $f(r)$ is defined, for instance, on $(0, \infty)$, then $L = \|\boldsymbol{v}_k\| = (\sum_{j=1}^{k} \lambda_j^2)^{1/2}$. The densities $\varrho_{M\beta}(\boldsymbol{v}_k)$ are reference k-eigenvalue densities of the corresponding FTE(β) ensembles. Equation (2.19) simply expresses the fact that the total density is the sum of rescaled densities $\varrho_{r,M\beta}(\boldsymbol{v}_k)$ associated with spheres of radii r, namely $\varrho_{r,M\beta}(\boldsymbol{v}_k) = \varrho_{M\beta}(\boldsymbol{v}_k/r)/r^k$, with varying weights $f(r)\,\mathrm{d}r$. For a given $\boldsymbol{v}_k$, only the densities $\varrho_{r,M\beta}(\boldsymbol{v}_k)$ with $r \geqslant \|\boldsymbol{v}_k\|$ contribute to the total density $\varrho_{S\beta}(\boldsymbol{v}_k)$ as

$$\varrho_{M\beta}(\boldsymbol{v}_k) = 0 \qquad \text{for} \quad \|\boldsymbol{v}_k\| > 1. \tag{2.21}$$

The relation between the density of states of the spherical ensemble $S_N(\beta)$ and that of FTE(β) is in particular found to be

$$\varrho_{S\beta}(\lambda) = \int_{|\lambda|}^{\infty} \varrho_{M\beta}\left(\frac{\lambda}{r}\right)\frac{f(r)}{r}\,\mathrm{d}r. \tag{2.22}$$

The distribution $\varrho_{M\beta}(\lambda)$ is even as both vectors $\boldsymbol{U}^{(N_p)}$ and $-\boldsymbol{U}^{(N_p)}$ are equally probable. In the case of complex spherical RMEs, (2.22) is replaced by a relation between radial densities (section 2.1) ($\varrho_{MC}(r) = 0$ for $r > 1$):

$$\varrho_{SC}(r) = \int_{r}^{\infty} \varrho_{MC}\left(\frac{r}{x}\right)\frac{f(x)}{x^2}\,\mathrm{d}x \qquad (r \geqslant 0). \tag{2.23}$$

Once $\varrho_{M\beta}(\lambda)$ and $\varrho_{MC}(r)$ are known, equations (2.22) and (2.23) yield the exact densities of states of the investigated spherical ensembles. The purpose of section 3 is thus to derive exact densities $\varrho_{M\beta}(\lambda)$ and $\varrho_{MC}(r)$ from those of the Gaussian ensembles of the same symmetry.

2.5. Monte Carlo simulations

Monte Carlo simulations were performed to generate matrices $M_N = \mathrm{Mat}(\boldsymbol{U}^{(N_p)})$ for real-symmetric, Hermitian and complex FTEs respectively. The random vector $\boldsymbol{U}^{(N_p)}$ was obtained from the stochastic representation, $\boldsymbol{U}^{(N_p)} \stackrel{\mathrm{d}}{=} \boldsymbol{G}^{(N_p)}/\|\boldsymbol{G}^{(N_p)}\|$ [22,31,32], where $\boldsymbol{G}^{(N_p)}$ is a N_p-dimensional vector whose components are identically and independently distributed $N(0,1)$ Gaussian random variables. The $\boldsymbol{G}^{(N_p)}$ components were generated by the classical Box–Müller method [33].

3. The density of states of FTE(β) and of FTE(c)

As emphasized in the introduction, Gaussian ensembles are notable members of the spherical family for which exact eigenvalue densities have been reported. It is the latter knowledge which is the basis of the following calculation.

By definition, a χ^2 distribution with n degrees of freedom is the law of the sum of the squares of n independent $N(0,1)$ Gaussian random variables [34]. The distributions of $z = \mathrm{tr}(S_N^2)/\sigma^2$ and of $z = 2\,\mathrm{tr}(S_N S_N^+)/\sigma^2$ are thus χ^2 distributions with N_p degrees of freedom for the GOE, GUE and complex Gaussian ensembles respectively. When applied to the Gaussian ensembles considered in the present paper, (2.17) reduces to

$$G_N \stackrel{\mathrm{d}}{=} RM_N \tag{3.1}$$

where G_N is a matrix from the GOE, the GUE or the complex Gaussian ensemble, R^2/σ^2 (GOE, GUE) and $2R^2/\sigma^2$ (complex case) are χ^2 distributed with N_p degrees of freedom and M_N is a matrix of the same symmetry as G_N associated with a vector $U^{(N_p)}$ which is uniformly distributed on the surface of the unit sphere in $I\mathbb{R}^{N_p}$. Equations (2.20), (2.22) and (2.23) (symbol S being replaced by G) then become for Gaussian ensembles

$$\varrho_{G\beta}(\lambda) = \frac{1}{2^{\frac{N_p}{2}-1}\sigma^{N_p}\Gamma(\frac{N_p}{2})}\int_{|\lambda|}^{\infty} x^{N_p-2}\exp\left(-\frac{x^2}{2\sigma^2}\right)\varrho_{M\beta}\left(\frac{\lambda}{x}\right)\mathrm{d}x \tag{3.2}$$

with N_p given by (2.1) and

$$\varrho_{GC}(r) = \frac{2}{(N^2-1)!\sigma^{2N^2}}\int_{r}^{\infty} x^{2N^2-3}\exp\left(-\frac{x^2}{\sigma^2}\right)\varrho_{MC}\left(\frac{r}{x}\right)\mathrm{d}x. \tag{3.3}$$

A general relation between the characteristic functions

$$\phi_{G\beta}(\boldsymbol{t}_k) = \langle\exp(\mathrm{i}\boldsymbol{v}_k\cdot\boldsymbol{t}_k)\rangle_{G\beta} \qquad (\boldsymbol{t}_k = (t_1, t_2, \ldots, t_k))$$

and

$$\phi_{M\beta}(\boldsymbol{t}_k) = \langle\exp(\mathrm{i}\boldsymbol{v}_k\cdot\boldsymbol{t}_k)\rangle_{M\beta}$$

is derived directly from (2.19) expressed for the Gaussian case:

$$\begin{aligned}\varrho_{G\beta}(\boldsymbol{v}_k) &= \frac{1}{C_{N\beta}}\int_{\|\boldsymbol{v}_k\|}^{\infty}\varrho_{M\beta}\left(\frac{\boldsymbol{v}_k}{r}\right)r^{N_p-1-k}\mathrm{e}^{-\frac{r^2}{2\sigma^2}}\,\mathrm{d}r\\ C_{N\beta} &= 2^{\frac{N_p}{2}-1}\sigma^{N_p}\Gamma\left(\frac{N_p}{2}\right)\end{aligned} \tag{3.4}$$

taking condition (2.21) into account. It is given by

$$\phi_{G\beta}(\boldsymbol{t}_k) = \frac{1}{C_{N\beta}}\int_0^{\infty}\phi_{M\beta}(r\boldsymbol{t}_k)r^{N_p-1}\mathrm{e}^{-\frac{r^2}{2\sigma^2}}\,\mathrm{d}r. \tag{3.5}$$

As already known [3], the densities of states of the Gaussian ensembles tend asymptotically to those of fixed-trace ensembles of the same symmetry (end of appendix A). As commented by Mehta [3, p 380], it is still difficult to know if all local statistical properties of the eigenvalues in the Gaussian and fixed-trace ensembles are asymptotically identical as n-level correlations of the FTEs, that are related to those of Gaussian ensembles by (3.4) and (3.5), are not yet explicitly known except for $n = 2$ given by equation (2.28) of [24]. However, the discussion of the universality of spectral fluctuations of spherical random matrix ensembles presented in [19] suggests that Gaussian ensembles and FTEs of a given symmetry belong to the same universality class.

Equations (3.2)–(3.5) for $k = 1$, can all be expressed as Laplace transforms:

$$p(s) = \int_0^{\infty}\exp(-sy)P(y)\,\mathrm{d}y = \mathsf{L}(P(y)) \tag{3.6}$$

by a change of variable which transforms $\exp(-\frac{x^2}{a\sigma^2})(a = 1 \text{ or } 2)$ into $\exp(-sy)$. The k-eigenvalues densities of the FTEs may also be obtained in theory, whatever k, from the corresponding densities of the Gaussian ensembles via inverse one-dimensional Laplace transforms. For $k = 1$ (3.5) becomes, for instance with $y = r^2t^2$ and $s = 1/(2\sigma^2t^2)$,

$$\Gamma\left(\frac{N_p}{2}\right)s^{-\frac{N_p}{2}}\phi_{G\beta}\left(\frac{1}{\sqrt{2s\sigma^2}}\right) = \int_0^{\infty}\phi_{M\beta}(\sqrt{y})y^{\frac{N_p}{2}-1}\mathrm{e}^{-sy}\,\mathrm{d}y \tag{3.7}$$

and thus

$$\begin{aligned}&p_{\phi\beta}(s) = \Gamma\left(\frac{N_p}{2}\right)s^{-\frac{N_p}{2}}\phi_{G\beta}\left(\frac{1}{\sqrt{2s\sigma^2}}\right)\\ &P_{\phi\beta}(y) = L^{-1}(p_{\phi\beta}(s)) \qquad \phi_{M\beta}(t) = t^{-(N_p-2)}P_{\phi\beta}(t^2)\end{aligned} \tag{3.8}$$

Table 1. Level density $\varrho_{M2}(\lambda)(-1 \leqslant \lambda \leqslant 1)$ for a FTE of Hermitian random matrices $M_N(2)$ for $2 \leqslant N \leqslant 6$ (see also [24]).

N	$\varrho_{M2}(\lambda)$
2	$\frac{1}{\pi\sqrt{(1-\lambda^2)}}$
3	$\frac{35}{64}(1-\lambda^2)(1-2\lambda^2+9\lambda^4)$
4	$\frac{256}{429\pi}(1-\lambda^2)^{7/2}(3+30\lambda^2-212\lambda^4+608\lambda^6)$
5	$\frac{2028\,117}{8388\,608}(1-\lambda^2)^7(3-12\lambda^2+898\lambda^4-5996\lambda^6+13\,555\lambda^8)$
6	$\frac{268\,435\,456}{583\,401\,555\pi}(1-\lambda^2)^{23/2}$ $\times(5+140\lambda^2-4020\lambda^4+70\,504\lambda^6-396\,536\lambda^8+774\,922\lambda^{10})$

where L^{-1} denotes the inverse Laplace transform. Similarly, taking $\lambda \geqslant 0(\varrho_{M\beta}(-\lambda) = \varrho_{M\beta}(\lambda))$ in (3.2), we obtain with condition (2.21), $y = x^2/\lambda^2$ and $s = \lambda^2/(2\sigma^2)$,

$$\begin{aligned} p_{\varrho\beta}(s) &= \sigma\sqrt{2}\Gamma\left(\frac{N_p}{2}\right)s^{-\frac{(N_p-1)}{2}}\varrho_{G\beta}\left(\sqrt{2s\sigma^2}\right) \\ P_{\varrho\beta}(y) &= L^{-1}(p_{M\beta}(s)) \qquad \varrho_{M\beta}(\lambda) = \lambda^{(N_p-3)}P_{\varrho\beta}\left(\frac{1}{\lambda^2}\right). \end{aligned} \tag{3.9}$$

In the complex case, the radial density $\varrho_{MC}(r)$ is finally calculated for $\sigma = 1$ from

$$\begin{aligned} p_{\varrho C}(s) &= (N^2-1)!s^{-(N^2-1)}\varrho_{GC}(\sqrt{s}) \\ P_{\varrho C}(y) &= L^{-1}(p_{\varrho C}(s)) \qquad \varrho_{MC}(r) = r^{2(N^2-2)}P_{\varrho C}\left(\frac{1}{r^2}\right) \end{aligned} \tag{3.10}$$

and from the density of the corresponding Gaussian ensemble (equation (2.13)). Appendix B gives relations between the non-zero moments of the Gaussian ensembles and those of the FTEs which are directly derived from equations (3.2) and (3.3). The Laplace transform method is first illustrated with FTE(2) and then applied to FTE(1).

3.1. FTE(2)

The density $\varrho_{M2}(\lambda)$ may be derived either from $\varrho_{G2}(\lambda)$ (equations (2.3) and (3.9)) or from an inversion of its cf $\phi_{M2}(t)$ which is obtained from $\phi_{G2}(t)$ (equations (2.5) and (3.8)). We have chosen the latter path to further obtain the unknown $\phi_{M2}(t)(\sigma = 1/\sqrt{2})$:

$$p_{\phi 2}(s) = \frac{1}{N}\Gamma\left(\frac{N^2}{2}\right)\left\{\sum_{m=0}^{N-1}\left[\frac{(-1)^m\binom{N}{m+1}\mathrm{e}^{-\frac{1}{4s}}}{2^m m!s^{\frac{N^2}{2}+m}}\right]\right\}.$$

From $L^{-1}(\frac{\mathrm{e}^{-a/s}}{s^{n+1}}) = (\frac{y}{a})^{n/2}J_n(2\sqrt{ay})(n > -1)$ (integral 6.643.4 of [28], where $J_n(x)$ is a Bessel function) we obtain

$$\phi_{M2}(t) = \frac{2^{\frac{N^2}{2}-1}}{N}\Gamma\left(\frac{N^2}{2}\right)\left\{\sum_{m=0}^{N-1}\left[\frac{(-1)^m\binom{N}{m+1}J_{\frac{N^2}{2}-1+m}(t)}{m!t^{\frac{N^2}{2}-1-m}}\right]\right\}. \tag{3.11}$$

We have further verified that the moments $\langle\lambda^{2k}\rangle_{M2}$ calculated from an expansion of $\phi_{M2}(t)$ agree with those obtained from (B.2). Integral 6.699.2 of [28] $(0 < \lambda^2/a^2 < 1)$,

$$\int_0^\infty t^\alpha J_\nu(at)\cos(\lambda t)\,\mathrm{d}t = \frac{2^\alpha\Gamma(\frac{1+\alpha+\nu}{2})}{\Gamma(\frac{\nu-\alpha+1}{2})}F\left(\frac{1+\alpha+\nu}{2}, \frac{1+\alpha-\nu}{2}, \frac{1}{2}; \frac{\lambda^2}{a^2}\right) \tag{3.12}$$

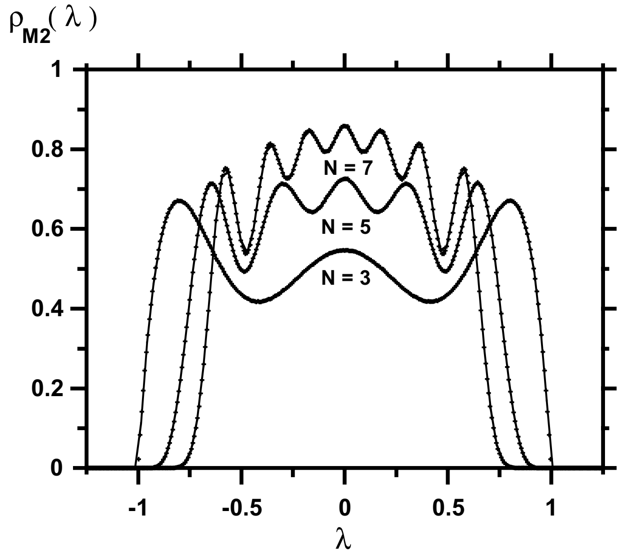

Figure 1. Density of states $\varrho_{M2}(\lambda)$ (equation (3.14) and table 1) for FTE(2) and $N = 3, 5, 7$ (crosses $= \varrho_{M2}(\lambda)$ calculated from Monte Carlo simulations with 5×10^7 matrices).

where $F(a, b, c; z)$ is a hypergeometric function, and equation (3.11) yield ($-1 \leqslant \lambda \leqslant 1$)

$$\varrho_{M2}(\lambda) = \frac{\Gamma(\frac{N^2}{2})}{N\pi\Gamma(\frac{N^2-1}{2})} \times \left\{ \sum_{m=0}^{N-1} \frac{(-1)^m 2^m}{m!} \binom{N}{m+1} \Gamma\left(m + \frac{1}{2}\right) F\left(m + \frac{1}{2}, \frac{3-N^2}{2}, \frac{1}{2}; \lambda^2\right) \right\}. \tag{3.13}$$

$F(m + \frac{1}{2}, \frac{3-N^2}{2}, \frac{1}{2}; \lambda^2)$ may be further expressed with the help of Gegenbauer polynomials $\mathsf{C}_{2m}^{\alpha}(\lambda)$ to yield, finally,

$$\varrho_{M2}(\lambda) = \frac{(\frac{N^2}{2} - 1)}{N\pi\Gamma(\frac{N^2-1}{2})}(1-\lambda^2)^{\frac{N^2-2N-1}{2}} \times \left\{ \sum_{m=0}^{N-1} (-1)^m \binom{N}{m+1} 2^m \Gamma(m + \tfrac{1}{2})(1-\lambda^2)^{N-1-m} \times \left[\sum_{j=0}^{m} \frac{(-1)^j (2\lambda)^{2j}}{(2j)!(m-j)!} \Gamma\left(\frac{N^2}{2} - 1 + j\right) \right] \right\}. \tag{3.14}$$

As explained in the introduction, the density, equation (3.14) (table 1 for $2 \leqslant N \leqslant 6$ and figure 1 for $N = 3, 5, 7$), is identical with the density published very recently by Akemann *et al* [24]. The density exhibits N local maxima for $-1 < \lambda < 1$. From (3.11), we deduce the asymptotic form of $\phi_{M2}(t)$, $\phi_{M2}(t) \sim (N^{1/2}/t)J_1(2t/N^{1/2})$, which is as expected the characteristic function of a Wigner semicircle of radius $a_\infty = 2/N^{1/2}$ (equation (2.12)) as $\langle\lambda^2\rangle_{M2} = 1/N = a_\infty^2/4$. The thermodynamic limit is thus obtained for a FTE(2) associated with a sphere of radius $R \propto N^{1/2}$, as found also for the GUE (appendix A). The Wigner semicircle is already an excellent numerical approximation of $\varrho_{M2}(\lambda)$ for N as small as 50.

Table 2. Level density $\varrho_{M1}(\lambda)(-1 \leqslant \lambda \leqslant 1)$ for a FTE of real-symmetric random matrices $M_N(1)$ for $2 \leqslant N \leqslant 5$.

N	$\varrho_{M1}(\lambda)$
2	$\frac{1}{2\sqrt{2}}\{I_{[0,\frac{1}{\sqrt{2}}[}(\lvert\lambda\rvert) + I_{[\frac{1}{\sqrt{2}},1[}(\lvert\lambda\rvert)\frac{\lvert\lambda\rvert}{\sqrt{(1-\lambda^2)}}\}$
3	$I_{[0,\frac{1}{\sqrt{2}}]}(\lvert\lambda\rvert)(\frac{16\sqrt{2}}{9\pi}(1-2\lambda^2)^{\frac{3}{2}}) + I_{[0,1]}(\lvert\lambda\rvert)[\frac{8}{9\pi}(1-\lambda^2)^{\frac{1}{2}}(7\lambda^2-1)]$
4	$I_{[0,\frac{1}{\sqrt{2}}]}(\lvert\lambda\rvert)\left(\frac{16\sqrt{2}}{35\pi}\frac{(1-2\lambda^2)^{5/2}}{(1-\lambda^2)}\left\{\begin{array}{c}3(1+11\lambda^2-12\lambda^4)\\ +70\lambda^4 S_2(\frac{4\lambda^2}{1-\lambda^2})\\ -7\lambda^2(1-2\lambda^2)S_3(\frac{4\lambda^2}{1-\lambda^2})\end{array}\right\}\right)$
	$+I_{[\frac{1}{\sqrt{2}},1]}(\lvert\lambda\rvert)[\frac{3}{\sqrt{2}}\lvert\lambda\rvert(5\lambda^2-1)(1-\lambda^2)^2]$
	with $S_p(z) = \sum_{k=0}^{\infty}\frac{(p+k)!(k+1)!}{(2k+2)!}z^k, \qquad p = 2, 3$
5	$\frac{3003}{10\,240}\left\{\begin{array}{l}I_{[0,\frac{1}{\sqrt{2}}]}(\lvert\lambda\rvert)(\sqrt{2}(1-2\lambda^2)^4(1+44\lambda^2+68\lambda^4))\\ +I_{[0.1]}(\lvert\lambda\rvert)[(1-\lambda^2)^4(1-50\lambda^2+209\lambda^4)]\end{array}\right\}$

3.2. FTE(1)

The expressions of the characteristic function and of the density of states are complicated in the real symmetric case. Expressions of Σ_{1t}, Σ_{2t}, Σ_{10}, Σ_{20}, S_{10}, S_{20}, S_{1E}, S_{2E1}, S_{2E2} which are needed to calculate them from (3.15)–(3.18) below are given in appendix C. A characteristic function $\phi_{M1}(t)$ which is valid whatever N and involves infinite sums can be derived from $\phi_{G1}(t)$ (equations (2.9) and (3.8)):

$$\phi_{M1}(t) = \frac{2^{\frac{N_p}{2}-1}}{N}\Gamma\left(\frac{N_p}{2}\right)\{\Sigma_{1t} - \Sigma_{2t}\}. \tag{3.15}$$

Simpler expressions can be calculated for $N = 2p + 1$ by inverting the characteristic function $\phi_{M1}(t)$ which is obtained from $\phi_{G1}(t)$ (equations (2.10) and (3.8)). Using the same method as in section 3.1, lengthy calculations yield the density $\varrho_{M1}(\lambda)$ from the expansion of Laguerre and Hermite polynomials in $\phi_{G1}(t)$ and from the inverse Laplace transform of $L^{-1}(\frac{e^{-a/s}}{s^{n+1}})$:

$$\phi_{M1}(t) = \frac{2^{\frac{N_p}{2}-1}}{N}\Gamma\left(\frac{N_p}{2}\right)\{\Sigma_{10} + \Sigma_{20}\}. \tag{3.16}$$

Expressing, as before, hypergeometric functions in terms of Gegenbauer polynomials, we finally obtain for $N = 2p + 1$ that

$$\varrho_{M1}(\lambda) = \frac{(\frac{N_p}{2}-1)}{N\pi\Gamma(\frac{N_p-1}{2})}(I_{[0,1]}(\lvert\lambda\rvert)\boldsymbol{S}_{10} + I_{[0,\frac{1}{\sqrt{2}}]}(\lvert\lambda\rvert)\boldsymbol{S}_{20}). \tag{3.17}$$

For $N = 2p (p \geqslant 2)$, the density $\varrho_{M1}(\lambda)$ is calculated directly via (3.9) from an inverse Laplace transform of the density $\varrho_{G1}(\lambda)$ (equations (2.6) and (2.8)):

$$\varrho_{M1}(\lambda) = I_{[\frac{1}{\sqrt{2}},1]}(\lvert\lambda\rvert)\boldsymbol{S}_{IE} + I_{[0,\frac{1}{\sqrt{2}}]}(\lvert\lambda\rvert)(\boldsymbol{S}_{2E1} + \boldsymbol{S}_{2E2}). \tag{3.18}$$

Densities of states are explicitly given in table 2 for small values of N (introduction), $2 \leqslant N \leqslant 5$, and are compared to simulated densities in figure 2. Appendix D discusses further the distributions of the eigenvalue spacings for FTE(β) and $N = 2$ as the distributions of spacings between successive eigenvalues of large matrices are well represented by those of ensembles of $N = 2$ matrices [38–41]. The underlying similarity of FTE(1) and of FTE(2) which originates from the unit sphere is more clearly seen on characteristic functions than it is on densities of states.

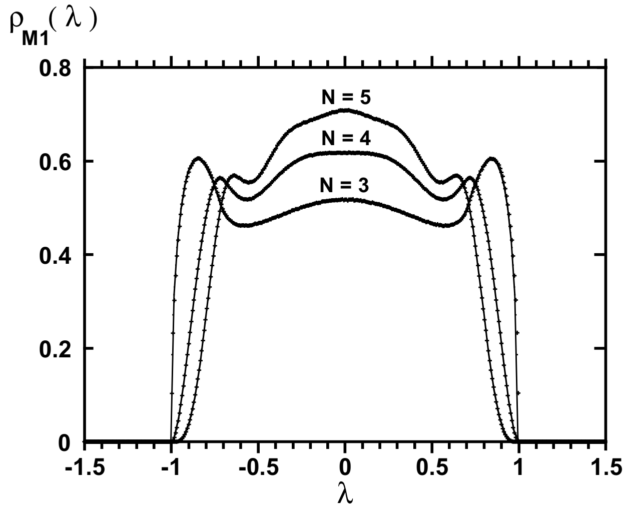

Figure 2. Density of states $\varrho_{M1}(\lambda)$ (equations (3.17) and (3.18) and table 2) for FTE(1) and $N = 3, 4, 5$ (crosses = $\varrho_{M1}(\lambda)$ calculated from Monte Carlo simulations with 5×10^7 matrices).

Table 3. Density $\varrho_{MC}(r)(0 \leqslant r \leqslant 1)$ for a FTE of complex random matrices $M_N(C)$ for $2 \leqslant N \leqslant 5$.

N	$\varrho_{MC}(r)$
2	$\frac{3}{2\pi}(1 - r^4)$
3	$\frac{8}{3\pi}(1 - r^2)^5(1 + 5r^2 + 15r^4)$
4	$\frac{15}{4\pi}(1 - r^2)^{11}(1 + 11r^2 + 66r^4 + 286r^6)$
5	$\frac{24}{5\pi}(1 - r^2)^{19}(1 + 19r^2 + 190r^4 + 1330r^6 + 7315r^8)$

3.3. FTE(c)

The radial density $\varrho_{MC}(r)$ is calculated from $\varrho_{GC}(r)$ ((2.13), $\sigma = 1$) using (3.10) with

$$p_{\varrho C}(s) = \frac{(N^2 - 1)!}{N\pi}\left[\sum_{k=0}^{N-1} \frac{e^{-s}}{k!s^{(N^2-1-k)}}\right].$$

From $L^{-1}(\frac{e^{-s}}{s^{n+1}}) = \frac{(y-1)^n}{n!}$ for integer $n \geqslant 0$ and $y > 1$ and with $y = 1/r^2$ (equation (3.10)), the FTE(c) radial density is finally given by $(0 \leqslant r < 1)$

$$\varrho_{MC}(r) = \frac{(N^2 - 1)}{N\pi}(1 - r^2)^{N^2-N-1}\left[\sum_{m=0}^{N-1}\binom{N^2 - N - 2 + m}{m} r^{2m}\right]. \quad (3.19)$$

The bracketed polynomial in the right of density (3.19) is recognized to be a truncated expansion of $(1 - r^2)^{-(N^2-N-1)}$ which ensures that $\varrho_{MC}(r)$ is asymptotically constant on a disc of radius $N^{-1/2}$ (section 2.1). Appendix B describes a direct calculation of the moments $\langle r^m \rangle_{MC}$. Table 3 gives $\varrho_{MC}(r)$ for $2 \leqslant N \leqslant 5$ and figure 3 compares theoretical radial densities to simulated ones for $N = 2, 3, 5$.

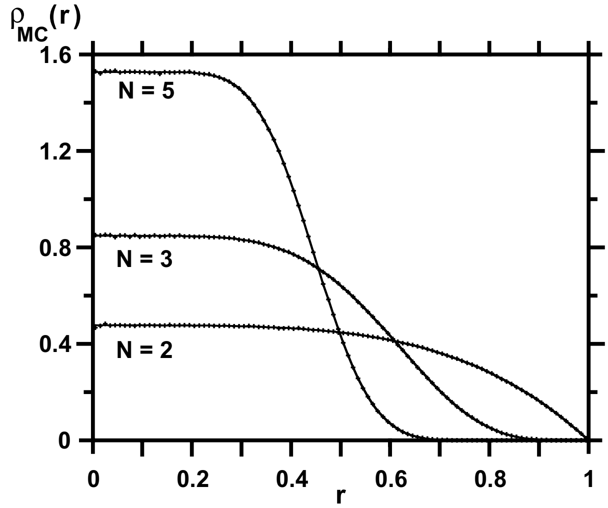

Figure 3. Radial density $\varrho_{MC}(r)$ (equation (3.19) and table 3) for FTE(c) and $N = 2, 3, 5$ (crosses $= \varrho_{MC}(r)$ calculated from Monte Carlo simulations with 5×10^7 matrices).

4. Some large-N densities of states

We describe further applications of the relation between the densities of spherical ensembles and those of the FTEs (equation (2.19)) to the calculation of asymptotic densities of states of some ensembles which were briefly considered in [19]. For large N, (2.22) becomes

$$\varrho_\infty(\lambda) = \lim_{N\to\infty} \int_{|\lambda|}^{\infty} \gamma_N r^{(N_p-3)} g\left(\frac{Nr^2}{4}\right) (r^2 - \lambda^2)^{1/2}\, \mathrm{d}r \tag{4.1}$$

with

$$\varphi_N = \int_0^\infty r^{(N_p-1)} g(r^2)\, \mathrm{d}r \qquad \gamma_N = N^{(N_p/2)}/(2^{(N_p-1)}\pi\varphi_N).$$

We consider first orthogonal (CLOE) and unitary (CLUE) Cauchy–Lorentz ensembles, whose associated $\mathbf{Vect}(H_N)$ are distributed according to a spherical Cauchy distribution [22, 34], with a characteristic function [35]

$$\phi(\boldsymbol{t}) = \langle \exp(\mathrm{i}\boldsymbol{t}\cdot\mathbf{Vect}(H_N))\rangle = \exp\left(-\gamma\left[\sum_{k=1}^{N_p} t_k^2\right]^{1/2}\right)$$

that is

$$p_\beta(H_N) = \frac{K_{\beta,N}}{\left(1+\frac{\mathrm{tr}(H_N^2)}{\gamma^2}\right)^{\frac{(N_p+1)}{2}}}. \tag{4.2}$$

Here $\gamma^2 = \alpha_G^2/N$, (α_G = constant), $g(r^2) = 1/(1+r^2/\gamma^2)^{(N_p+1)/2}$. The marginal distribution of every distinct matrix element (section 2) is thus a Cauchy distribution:

$$f_\beta(B_{ij}) = \frac{\pi\sqrt{(2-\delta_{ij})}}{4\gamma}\frac{1}{1+\frac{B_{ij}^2}{\gamma^2}(2-\delta_{ij})} \tag{4.3}$$

where B_{ij} is the matrix element H_{ij} for $\beta = 1$ while it is either a diagonal element or the real or imaginary part of a non-diagonal element for $\beta = 2$. The Lévy ensembles of random matrices [36] with $\alpha = 1$ have also distinct matrix elements (with characteristic function $\phi_\alpha(t) = \exp(-\gamma|t|^\alpha)$) which are Cauchy distributed but they are independently distributed in contrast to those considered here. Consequently, the distribution of any linear combination of $k(k > 1)$ non-diagonal matrix elements with equal weights $k^{-1/2}$ is still given by (4.3) as deduced from the characteristic function $\phi(t)$ (see (2.14) and appendix A). For independent entries with distribution (4.3), the latter linear combination would have a broader Cauchy distribution as γ would be multiplied by $k^{1/2}$. Equation (4.1) yields [19]

$$\varrho_{\beta,1,\infty}(\lambda) = C_1\left[1 - \int_0^{\frac{\pi}{2}} \cos(\theta)\exp(-y(\lambda,\theta))\,\mathrm{d}\theta\right] \tag{4.4}$$

$$y(\lambda,\theta) = \beta\alpha_G^2\frac{\cos^2(\theta)}{\lambda^2} \tag{4.5}$$

with $C_1 = 2\pi^{-3/2}\beta^{-1/2}\alpha_G^{-1}$. For small λ, distribution (4.4) has a parabolic variation and is flatter than a Lorentz line whose maximum density is chosen as C_1. It decreases as λ^{-2} as $\lambda \to \pm\infty$ and has diverging moments.

Using (4.1), density (4.4) is generalized to orthogonal or unitary Student's [22] RMEs for which the exponent of the Cauchy distribution $(N_p + 1)/2$ (4.2) is replaced by $(N_p + m)/2$ (and $K_{\beta,N}$ by $K_{\beta,m,N}$) where m is an integer ($m \geqslant 1$):

$$\varrho_{\beta,m,\infty}(\lambda) = C_m\int_0^{\frac{\pi}{2}}\left(\frac{\cos^m(\theta)}{|\lambda|^{m+1}}\exp(-y(\lambda,\theta))\sin^2(\theta)\right)\mathrm{d}\theta \tag{4.6}$$

($C_m = 4\beta^{m/2}\alpha_G^m/(\pi\Gamma(m/2))$ where α_G can be chosen to depend on m). When $m = 2k + 1$, (4.6) may also be expressed as

$$\varrho_{\beta,2k+1,\infty}(\lambda) = C_{2k+1}\left[1 - \int_0^{\frac{\pi}{2}}\cos(\theta)\exp(-y(\lambda,\theta))e_k(y(\lambda,\theta))\,\mathrm{d}\theta\right] \tag{4.7}$$

$$e_k(x) = \sum_{j=0}^{k}\frac{x^j}{j!} \tag{4.8}$$

($C_{2k+1} = 2^{2k+1}\pi^{-3/2}\beta^{-1/2}\alpha_G^{-1}(k!)^2/(2k!)$). The density $\varrho_{\beta,m,\infty}(\lambda)$ varies as $|\lambda|^{-(m+1)}$ when $\lambda \to \pm\infty$ and parabolically for small values of λ. When m increases, the densities evolve progressively to the Wigner semicircle as shown by figure 4. When $\beta^{1/2}\alpha_G$ is chosen so that $C_{2k+1} = (2/\pi)^{3/2}$, the even moments of $\varrho_{\beta,2k+1,\infty}(\lambda)$ are indeed ($n \leqslant k$):

$$\langle\lambda^{2n}\rangle = (2n)!(2k-2n)!(k!)^{4n+1}2^{4nk-n}/[n!(n+1)!(k-n)!((2k)!)^{2n+1}]. \tag{4.9}$$

The latter moments tend for large k to $\langle\lambda^{2n}\rangle_\infty = (\pi/8)^n(2n)!/[n!(n+1)!]$ which coincide with the corresponding moments of the Wigner semi circle of radius $a_\infty = (\pi/2)^{1/2}$.

5. Conclusions

Exact expressions have been obtained respectively for the densities of states and the radial density of real-symmetric, Hermitian and complex ensembles of random matrices associated with a uniform distribution of a random vector on the surface of a N_p-dimensional unit sphere. The exact density $\varrho_S(\lambda)$ of any spherical ensemble of $N \times N$ random matrices of a given symmetry can be finally obtained from the $\varrho_M(\lambda)$ of the FTE of the same symmetry by a single one-dimensional integration involving ϱ_M and $f(r)$ (equation (2.20)), where $f(r)\,\mathrm{d}r$ is the probability of finding a N_p-dimensional sphere of radius $r \leqslant R \leqslant r + \mathrm{d}r$.

The method used in the present work might as well be extended to other RMEs, for instance to the symplectic ensemble FTE(4) or the ensemble of antisymmetric Hermitian matrices [3].

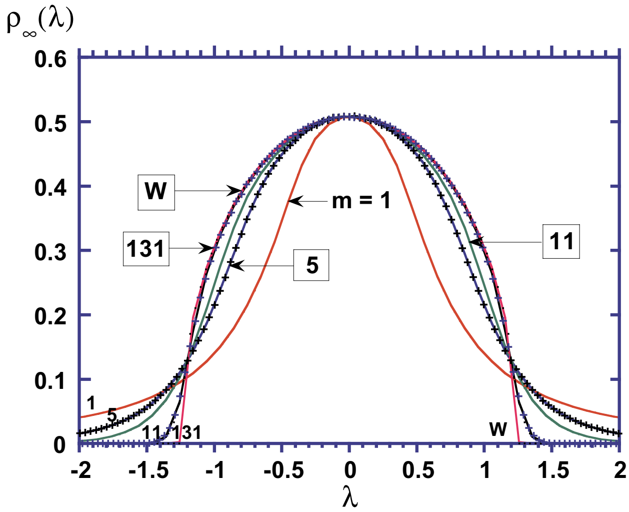

Figure 4. Asymptotic density of states $\varrho_\infty(\lambda)$ (equation (4.7)) for the unitary Student's RMEs with odd values of m: $m = 1$ (Cauchy–Lorentz ensemble [19]), $m = 5, 11, 131$ and $W =$ Wigner semicircle $\varrho_w(\lambda) = (4/\pi^2)(\pi/2 - \lambda^2)^{1/2}(m = \infty)$ (crosses = results obtained from Monte Carlo simulations with 5×10^7 matrices).

Other characterizations of the FTEs are desirable, for instance the determination of exact k-eigenvalue correlations as done for the unitary ensemble and $k = 2$ by Akemann *et al* [24].

Determinant distributions of FTEs have been derived from methods similar to those described here [25]. Exact results for FTEs may be of interest in relation to the probability distributions of various physical properties of disordered solids.

The asymptotic behaviour indicates that the matrices associated with 'typical' unit vectors, whose extremities are taken at random on the surface of the N_p-sphere, have Wigner semicircular densities for large N. The unit sphere surface provides a unifying way of considering spherical random matrix ensembles of different symmetries. For large N, both Gauss and Wigner distributions are simply retrieved from the unit sphere surface.

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Appendix A. Uniform distribution on the surface of the unit sphere in $I\mathbb{R}^{N_p}$

The joint distribution of $n(<N_p)$ components, $U(i), i = 1, \ldots, n$, of a unit vector $\boldsymbol{U}^{(N_p)}$ which is uniformly distributed on the surface of the unit sphere in $I\mathbb{R}^{N_p}$ is [22]

$$g_{N_p}(U(1), \ldots, U(n)) = \frac{\Gamma(N_p/2)}{\Gamma((N_p - n)/2)\pi^{n/2}} \left(1 - \sum_{i=1}^{n} U(i)^2\right)^{\frac{N_p-n-2}{2}} \qquad \text{(A.1)}$$

$$\sum_{i=1}^{n} U(i)^2 \leqslant 1$$

where $\Gamma(x)$ is the usual gamma function. Distribution (A.1) is, for instance, derived from the definition of a Dirichlet distribution [22] and from the fact that $\boldsymbol{U}^{(N_p)} \stackrel{d}{=} \boldsymbol{G}^{(N_p)}/\|\boldsymbol{G}^{(N_p)}\|$ [22,32], where $\boldsymbol{G}^{(N_p)}$ is a N_p-dimensional vector whose components are identically and independently distributed $N(0, 1)$ Gaussian random variables.

For a spherical random vector $\boldsymbol{X}^{(N_p)}$, the distribution of every linear form $y = \boldsymbol{a} \cdot \boldsymbol{X}^{(N_p)}$ is the same for all vectors $\boldsymbol{a}$, where $\boldsymbol{a} = (a_1 \ldots, a_{N_p})$, provided that $S = \sum_{i=1}^{N_p} a_i^2 = 1$ [22, p 31], [32, p 51] as seen, for instance, from the characteristic function $\langle e^{ity}\rangle = \langle e^{i\boldsymbol{a}\cdot\boldsymbol{X}^{(N_p)}}\rangle = \phi(|t|\sqrt{S}) = \phi(|t|)$ (equation (2.14)) whatever the a_i's. The distribution of $y = (U(j) + U(k))/\sqrt{2}$ $(j \neq k)$ needed in appendix D is thus identical with the distribution of any component $U(l)$: $y \stackrel{d}{=} U(l)$. Similarly, the distribution of $\mathrm{tr}(M_N(\beta))/N^{1/2}(M_N = \mathrm{Mat}(\boldsymbol{U}^{(N_p)}))$ is given by (A.1) with $n = 1$.

Expressing $M_N(\beta)^2$ in terms of the components of the unit vector $\boldsymbol{U}^{(N_p)}(\beta)$, squaring all matrix elements, summing up and averaging using distribution (A.1), we derive, independently from the calculation presented in section 3, the fourth moment $\langle\lambda^4\rangle_{M\beta} = \int_{-1}^{+1} \varrho_{M\beta}(\lambda)\lambda^4 \mathrm{d}\lambda$ as

$$\langle\lambda^4\rangle_{M1} = (2N^2 + 5N + 5)/[N(N + 1)(N^2 + N + 4)]$$

and

$$\langle\lambda^4\rangle_{M2} = (2N^2 + 1)/[N^2(N^2 + 2)]$$

for $\beta = 1$ and 2 respectively. The second moment, $\langle\lambda^2\rangle_{M\beta} = 1/N$, follows immediately from $\mathrm{tr}(M_N^2(\beta)) = 1$. The ratio: $\lim_{N\to\infty}\langle\lambda^4\rangle_{M\beta}/\langle\lambda^2\rangle_{M\beta}^2 = 2$ as expected for a Wigner semicircle (equation (2.11)) which is the asymptotic eigenvalue density of the FTE(β). A simple argument is that the GOE and GUE, whose asymptotic densities are Wigner semicircles of radii $a_{\beta\infty}$, tend for large N to FTEs, as $z = \mathrm{tr}(S_N^2)/\sigma^2$ has a chi-square distribution with N_p degrees of freedom which results in $\langle(z - \langle z\rangle)^2\rangle^{1/2}/\langle z\rangle \propto N^{-1}$. For large N, the radius of the associated sphere is thus $R = \sigma\langle\mathrm{tr}(S_N^2)\rangle^{1/2} = \sigma N(\beta/2)^{1/2}$. The second moment $\langle\lambda^2\rangle_{G\beta} = \sigma^2 N\beta/2 = a_{\beta\infty}^2/4$ gives then the usual scaling $\sigma \propto N^{-1/2}$, that is $R \propto N^{1/2}$, for the GOE and the GUE.

Appendix B. Moments of the FTEs

Relations between the non-zero moments of the Gaussian ensembles and those of the FTEs are directly derived from equations (3.2) and (3.3). The moments

$$\langle\lambda^{2k}\rangle_{X\beta} = \int_{-A}^{+A} \varrho_{X\beta}(\lambda)\lambda^{2k}\,\mathrm{d}\lambda \tag{B.1}$$

where $A = \infty, 1$ for $X = G, M$ respectively, satisfy

$$\langle\lambda^{2k}\rangle_{M\beta} = \langle\lambda^{2k}\rangle_{G\beta} \Bigg/ \left[2^k\sigma^{2k}\prod_{i=1}^{k}\left(\frac{N_p}{2} + i - 1\right)\right] \tag{B.2}$$

In the complex case, the moments of $p_{GC}(r) = 2\pi r\varrho_{GC}(r)$ and of $p_{MC}(\boldsymbol{r}) = 2\pi\boldsymbol{r}\varrho_{MC}(\boldsymbol{r})$ are related through (3.3):

$$\langle\boldsymbol{r}^m\rangle_{MC} = \langle r^m\rangle_{GC}\left(\frac{(N^2 - 1)!}{\sigma^m\Gamma(N^2 + \frac{m}{2})}\right). \tag{B.3}$$

Using (2.15) with $\sigma = 1$ and [37], we obtain

$$\langle\boldsymbol{r}^m\rangle_{GC} = \frac{1}{N}\sum_{k=0}^{N-1}\frac{1}{k!}\Gamma\left(k + \frac{m}{2} + 1\right) = \frac{\Gamma(\frac{m}{2} + 1)}{N}\sum_{k=0}^{N-1}\binom{k + \frac{m}{2}}{k}.$$

That is,

$$\langle \boldsymbol{r}^m \rangle_{GC} = \frac{\Gamma(\frac{m}{2}+1)}{N} \binom{N+\frac{m}{2}}{N-1} = \frac{2}{(m+2)} \frac{\Gamma(N+\frac{m}{2}+1)}{N!}$$

and thus

$$\langle \boldsymbol{r}^m \rangle_{MC} = \left(\frac{2}{m+2}\right) \left(\frac{(N^2-1)!}{\Gamma(N^2+\frac{m}{2})}\right) \left(\frac{\Gamma(N+\frac{m}{2}+1)}{N!}\right). \tag{B.4}$$

For a constant asymptotic density $\varrho_{GC\infty}(r) = N/\pi$ for $r \leqslant N^{-1/2}$ (section 3.3) and 0 otherwise, the moments are indeed $\langle r^m \rangle_{MC\infty} = \int_0^{N^{-1/2}} 2N\boldsymbol{r}^{m+1}\,\mathrm{d}r = \frac{2}{(m+2)N^{m/2}}$ as consistently found from (B.4) for large N.

Appendix C. Characteristic function $\phi_{M1}(t)$ and density of states $\varrho_{M1}(\lambda)$ of FTE(1)

The expressions needed to calculate the characteristic function $\phi_{M1}(t)$, and the density of states $\varrho_{M1}(\lambda)$ (equations (3.15)–(3.18)) are given below. The characteristic function $\phi_{M1}(t)$ given by (3.17) is obtained from

$$\Sigma_{1t} = \sum_{m=0}^{N-1} \left[\frac{(-1)^m \binom{N}{m+1} 2^{\frac{N_p}{4}-\frac{1}{2}-\frac{m}{2}}}{m!} \times \frac{\boldsymbol{J}_{\frac{N_p}{2}-1+m}(\frac{t}{\sqrt{2}})}{t^{\frac{N_p}{2}-1-m}} \right]$$

and

$$\Sigma_{2t} = \left(\frac{\Gamma(\frac{N+1}{2})(N-1)!}{\Gamma(\frac{N}{2})}\right)^{\frac{1}{2}} \sum_{m=0}^{\infty} \left[(-1)^m \varpi_{N,m} 2^{\frac{N_p}{4}-\frac{m}{2}-1} \times \frac{\boldsymbol{J}_{\frac{N_p}{2}+m}(\frac{t}{\sqrt{2}})}{t^{\frac{N_p}{2}-m-2}} \right]$$

with

$$\varpi_{N,m} = \sum_{k=0}^{n} \frac{\binom{N+2m-2k+1}{N-1-k}}{k!} \left(\frac{\Gamma(m-k+1+\frac{N}{2})}{\Gamma(m-k+\frac{N+3}{2})(2m-2k+N+1)!}\right)^{\frac{1}{2}}$$

($n = \frac{m+N-1}{2} - |\frac{m-N+1}{2}| = \min(m, N-1)$).

For N odd, $\phi_{M1}(t)$ is obtained from (3.16) with

$$\Sigma_{10} = \sum_{m=0}^{p} \left[\frac{(-1)^m \binom{p}{m} \sqrt{\pi}}{\Gamma(m+\frac{1}{2})} \times \frac{\boldsymbol{J}_{\frac{N_p}{2}-1+m}(t)}{t^{\frac{N_p}{2}-1-m}} \right]$$

and

$$\Sigma_{20} = \sum_{m=0}^{2p-2} \left[(-1)^m \alpha_{p,m} 2^{\frac{N_p}{4}-\frac{1}{2}-\frac{m}{2}} \times \frac{\boldsymbol{J}_{\frac{N_p}{2}-1+m}(\frac{t}{\sqrt{2}})}{t^{\frac{N_p}{2}-1-m}} \right]$$

with

$$\alpha_{p,m} = \frac{\binom{2p+1}{m+1}}{m!} - \sum_{k=0}^{p-|p-m|} \left[\frac{\Gamma(p-k+\frac{1}{2}) \binom{2p}{m+k}}{2^k \Gamma(p+\frac{1}{2})(m-k)!} \right].$$

For $N = 2p + 1$, the density $\varrho_{M1}(\lambda)$ is calculated from (3.17) with

$$S_{10} = \sqrt{\pi}\, p! \left\{ \sum_{m=0}^{p} (-2)^m \frac{(1-\lambda^2)^{\frac{N_p-3}{2}-m}}{(p-m)!} \times \left[\sum_{j=0}^{m} \frac{(-1)^j (2\lambda)^{2j} \Gamma(\frac{N_p}{2} - 1 + j)}{(2j)!(m-j)!} \right] \right\}$$

and

$$S_{20} = \left\{ \sum_{m=0}^{2p-2} (-1)^m 2^{m+1/2} \Gamma(m + \tfrac{1}{2}) m! \alpha_{p,m} (1 - 2\lambda^2)^{\frac{N_p-3}{2}-m} \right.$$
$$\left. \times \left[\sum_{j=0}^{m} \frac{(-1)^j 2^{3j} \lambda^{2j} \Gamma(\frac{N_p}{2} - 1 + j)}{(2j)!(m-j)!} \right] \right\}$$

while $\varrho_{M1}(\lambda)$ is calculated from (3.18) for $N = 2p (p \geqslant 2)$ with

$$S_{1E} = \frac{(-1)^p \Gamma(p + \frac{1}{2}) \Gamma(\frac{N_p}{2})}{N\sqrt{2\pi}} \times \left\{ \sum_{m=1}^{p} (-1)^m \frac{2^{3m-1} |\lambda|^{2m-1} (1-\lambda^2)^{\frac{N_p}{2}-m-1}}{(p-m)!(2m-1)!\Gamma(\frac{N_p}{2} - m)} \right\}$$

and

$$S_{2E1} = \frac{\sqrt{2}(\frac{N_p}{2} - 1)}{N\pi\Gamma(\frac{N_p-1}{2})} \left\{ \sum_{m=1}^{2p-3} (-1)^m \Gamma(m + \tfrac{1}{2}) 2^m (1 - 2\lambda^2)^{\frac{N_p-3}{2}-m} \beta_{p,m} \Sigma_m(\lambda) \right\}$$

$$\Sigma_m(\lambda) = \sum_{k=0}^{m} (-1)^k \frac{\Gamma(\frac{N_p}{2} - 1 + k) 2^{3k} \lambda^{2k}}{(m-k)!(2k)!}$$

$$\beta_{p,m} = \binom{2p}{m+1} - \Sigma_{k=0}^{p-\frac{1}{2}-|p-\frac{1}{2}-m|} \left[\frac{(p-1-k)!m! \binom{2p-1}{m+k}}{2^k (p-1)!(m-k)!} \right].$$

Finally,

$$S_{2E2} = \frac{(-1)^p \Gamma(p + \frac{1}{2}) \Gamma(\frac{N_p}{2}) \sqrt{2}}{2N\pi(1-\lambda^2)}$$
$$\times \left\{ \sum_{m=1}^{p} (-1)^m \frac{2^{3m} \lambda^{2m} (1 - 2\lambda^2)^{\frac{N_p-1}{2}-m}}{(p-m)!(2m-1)!\Gamma(\frac{N_p-1}{2} - m)} \right.$$
$$\left. \times F\left(1, \frac{N_p}{2} - m, \frac{3}{2}; \frac{\lambda^2}{1-\lambda^2}\right) \right\}$$

where

$$F\left(1, \frac{N_p}{2} - m, \frac{3}{2}, \frac{\lambda^2}{1-\lambda^2}\right) = \frac{2}{(\frac{N_p}{2} - m - 1)!} \sum_{k=0}^{\infty} \frac{(\frac{N_p}{2} - m - 1 + k)!(k+1)!}{(2k+2)!} \left(\frac{4\lambda^2}{1-\lambda^2}\right)^k$$

Appendix D. Spacing distributions for the FTE(β) ensembles with $\beta = 1, 2$, $N = 2$

The distributions of level spacings of large matrices are known to be well approximated by those of ensembles of $N = 2$ matrices [38–41]. The small spacing behaviour of Gaussian ensembles, $p(s) \propto s^\beta$, can indeed be reproduced by considering 2×2 matrices as a very close encounter of two levels is only weakly influenced by other levels [40, 41]. The eigenvalue spacing $p_2(s)$ of matrices $M_2(\beta)$ (equation (2.18)) is $s = \sqrt{(2 - x^2)}$ where $x = U(1) + U(2 + \beta) (-\sqrt{2} \leqslant x \leqslant \sqrt{2})$.

From $x \stackrel{\mathrm{d}}{=} U(1)\sqrt{2}$ and (A.1) with $n = 1$ and $N_p = 2 + \beta$, the distribution of x is found to be uniform for $\beta = 1$, $h(x) = 1/(2\sqrt{2})$, while $h(x) = \sqrt{(2 - x^2)}/\pi$ for $\beta = 2$.

The distribution $p_2(s)(c_1 = 1/\sqrt{2}, c_2 = 2/\pi)$:

$$p_2(s) = c_\beta s^\beta \Big/ \sqrt{(2 - s^2)} \qquad 0 \leqslant s \leqslant \sqrt{2} \tag{D.1}$$

yields further the spacing distribution for spherical ensembles with densities $g(\mathrm{tr}(S_N^2))(N = 2)$:

$$p_2(s) \propto s^\beta \int_{\frac{s^2}{2}}^{\infty} \frac{g(x)}{\sqrt{x - \frac{s^2}{2}}}\, \mathrm{d}x. \tag{D.2}$$

Distribution (D.1) varies as s^β for small s as do distributions (D.2) when the integral in (D.2) converges.

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